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On existence and uniqueness of solutions of mathematical model of biological control of cassava pests

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ABSTRACT

Mathematical model of biological control of two major pests of cassava was established using prey-predator approach. The existence and uniqueness of the model solution was verified through optimal control approach, and finally, the numerical method of Runge Kutta was employed to test the validity of the model using numerical data.

Keywords: prey-predator, exponential growth/decay, Optimal control, Injury level, Natural enemies.

1. Introduction

Pests are living organisms (plants or animals) that are injurious or cause loss or irritation to either plant or animal. Most farmers employ the use of chemical insecticides which has side effects on plants, humans and the environment at large in controlling pest, Aderinto et.al. (2013), Micheal (2008), Andres *et al.* (1979). And because of these harzard effects,there is need for alternative methods of controlling pest.

Biological control involves the use of parasitoids, predators and pathogens to maintain the population of pest at a level lower than it would without natural enemies. Some reseachers have studied the mathematical model as well as biological control of insect pests. Aderinto et.al. (2013) qualitatively studied biological pest control. Mustiya *et al.* (2014) worked on development and reproduction of cassava pests at different temperature. Picart *et. al.*(2011) studied optimal control of insect pest populations. Rafikov and Balthazar (2005), and Rafikov *et al.* (2008) studied mathematical modelling and control of pest population. Goh et.al (1977) worked on the optimal control of prey predator system. Herren and Neuenschwander(1991) looked at the biological control of cassava pest, to mention a few.

However, the present effort is to consider the the existence and uniqueness of the solution of the mathematical model of biological control of two major pests of cassava, called green spider mite (Mononychellus tanajoa) and cassava mealybug (Phenacoccus manihoti) through the use of their predators known as the predatory mites namely Typhlodromalus aripo and Epidinocarsis lopezi respectively. The optimallity system for the biological control of cassava pests was established, the uniqueness and existence solution to the system was verified, and finally numerical application was presented in an attempt to minimize the pest population below injury level and maximize cassava output.

2.Mathematical Model

Let the population of the prey and predator species respectively be represented by $M_1(t)$, $M_2(t)$ and $T_1(t)$, $T_2(t)$. As presented in the flow diagrams figures 2.1, 2.2a, and 2.2b.

The following assumptions were made:

i. The prey has unlimited supply of food.
ii. The predators depends completely on its prey as the only source of food and
iii. Each prey has no other threats except for its predator being studied

***Figure 2.1**: Flow diagram for the model*

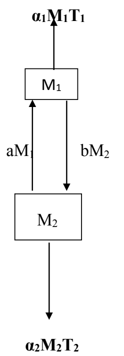


***Figure 2.2a:** Flow diagram for the model*

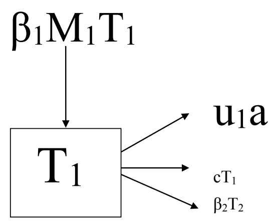
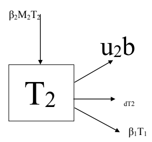


***Figure 2.2b:** Flow diagram for the model*



With the assumption that the pray have unlimited food supply, when there are no predator, the population of prey grows rapidily(exponential growth) and because of the limited time scale, we have

$$\frac{dM_1}{dt} = aM_1. \qquad (1)$$

and

$$\frac{dM_2}{dt} = bM_2. \qquad (2)$$

for the two species respectively, where $a, b > 0$, $T_1 = T_2 = 0$.
Also, the predator dies out exponenially in the absence of the prey(exponential decay), thus

$$\frac{dT_1}{dt} = -cT_1. \qquad (3)$$

and

$$\frac{dT_2}{dt} = -dT_2. \qquad (4)$$

where $c, d > 0$, $M_1 = M_2 = 0$.

The outcome of interactions between prey and predator are said to be proportional to the product of their population. And this in turn leads to increase in number, size, value and strength of the predator by βMT, and contrary for the prey by αMT with respect to each of the species.
Thus, the model is mathematically represented as;

$$
\begin{aligned}
\frac{dM_1}{dt} &= aM_1 - \alpha_1 M_1 T_1 - bM_2, \\
\frac{dM_2}{dt} &= bM_2 - \alpha_2 M_2 T_2 - aM_1, \\
\frac{dT_1}{dt} &= -cT_1 + \beta_1 M_1 T_1 - \beta_2 T_2 - u_1 a, \\
\frac{dT_2}{dt} &= -dT_2 + \beta_2 M_2 T_2 - \beta_1 T_1 - u_2 b.
\end{aligned} \tag{5}
$$

Parameters used are defined in table 2.1.

Table 2.1 : Definition of Parameters (Rafikof et al.2008 and Aderinto et al 2013)

Symbols	Definitions	
M_1, M_2	Population of prey species 1 and 2 respectively	To be observed
T_1 , T_2	Population of predator species 1 and 2 respectively	To be observed
a , b	birth rate of prey specie 1 and 2 respectively	0.17,0.116.
c , d	death rate of predator specie 1and 2 respectively	0.00017.
α_1 , α_2	death rate per interaction of prey species 1 and 2 respectively(predator attack rate)	0.20,0.20.
β_1 , β_2	growth rate of predators species 1and 2 respectively	0.0085,0.0085.
u_1 , u_2	control rate for species 1 and 2 respectively	$0 \le u_1 \le 1, \quad 0 \le u_2 \le 1.$

Optimality System

The optimality system is established with the aim of minimizing the pest attack below injury level and maximizing the cassava output. The optimal control problem is stated as
Minimize

$$J(u) = \int_{t_0}^{t_f} \left(CM_i + Au_i^2 \right) dt, \quad i = 1,2, \; 0 \le u_i \le 1. \tag{6}$$

Where the biological control associated cost is denoted by A, the control is denoted by μ: $0 \le \mu \le 1$, for effective result. C is the constant that characterize the weight assciated with prey. t0 and tf are the initial and final time of the control application respectively.
subject to the constraints

$$
\begin{aligned}
\frac{dM_1}{dt} &= aM_1 - \alpha_1 M_1 T_1 - bM_2, \\
\frac{dM_2}{dt} &= bM_2 - \alpha_2 M_2 T_2 - aM_1, \\
\frac{dT_1}{dt} &= -cT_1 + \beta_1 M_1 T_1 - \beta_2 T_2 - u_1 a. \\
\frac{dT_2}{dt} &= -dT_2 + \beta_2 M_2 T_2 - \beta_1 T_1 - u_2 b.
\end{aligned} \tag{7}
$$

with $M_i(t) = M_0, \quad T_i(t) = T_0,$ for $i = 1,2.$

3.1 Uniqueness of the System

Lemma 3.1 If p_i and q_i (i = 1,2,..., n)are Lipschitz continuous, then $\exists$ r, s such that

$$|p_i(u) - p_i(v)| \le r_i |u - v|, |q_i(u) - q_i(v)| \le s_i |u - v|. \tag{8}$$

$\forall u, v \in R$ and $i = 1,2,\cdots,n$. (Barbu 1994)

To estabilish the uniqueness of the solution to (7),
let ;

$$
\begin{aligned}
f_1 &= aM_1 - \alpha_1 M_1 T_1 - bM_2, \\
f_2 &= bM_2 - \alpha_2 M_2 T_2 - aM_1, \\
f_3 &= -cT_1 + \beta_1 M_1 T_1 - \beta_2 T_2 - u_1 a, \\
f_4 &= -dT_2 + \beta_2 M_2 T_2 - \beta_1 T_1 - u_2 b.
\end{aligned} \tag{9}
$$

Theorem 3.1
The system (9) has a unique solution in the region $0 \leq \mu \leq 1$, if the partial derivatives are continuous and bounded in the region.

Proof
Differentiate (9) partially with respect to M_1, M_2, T_1, T_2.

$$
\begin{aligned}
&\frac{\partial f_1}{\partial M_1} = |a - \alpha_1 T_1| < \infty && \frac{\partial f_1}{\partial M_2} = |-b| < \infty. \\
&\frac{\partial f_1}{\partial T_1} = |-\alpha_1 M_1| < \infty && \frac{\partial f_1}{\partial T_2} = 0 < \infty. \\
&\frac{\partial f_2}{\partial M_1} = |-a| < \infty && \frac{\partial f_2}{\partial M_2} = |b - \alpha_2 T_2| < \infty. \\
&\frac{\partial f_2}{\partial T_1} = 0 < \infty && \frac{\partial f_2}{\partial T_2} = |-\alpha_2 M_2| < \infty. \\
&\frac{\partial f_3}{\partial M_1} = |\beta_1 T_1| < \infty && \frac{\partial f_3}{\partial M_2} = 0 < \infty. \\
&\frac{\partial f_3}{\partial T_1} = |-c + \beta_1 M_1| < \infty && \frac{\partial f_3}{\partial T_2} = |\beta_2| < \infty. \\
&\frac{\partial f_4}{\partial M_1} = 0 < \infty && \frac{\partial f_4}{\partial M_2} = |\beta_1 T_2| < \infty. \\
&\frac{\partial f_4}{\partial T_1} = |-\beta_1| < \infty && \frac{\partial f_4}{\partial T_2} = |-d + \beta_2 M_2| < \infty.
\end{aligned}
$$

Since all the partial derivatives exist, that shows they are finite and bounded, hence there exist a unique solution.

The control space $U = u : u$ are measurable, $0 \leq u(t) \leq 1, t \in [t_0, t_f]$, convex and closed. The solutions to the state differential equations exist as well as bounded and are independent of the control. The Hamiltonian is defined as

$$H(t, M, T, u) = \sum_{i=1}^{2} (cM_i + Au_i^2) + p_1 \dot{M}_1 + p_2 \dot{M}_2 + p_3 \dot{T}_1 + p_4 \dot{T}_2. \tag{10}$$

Where p_1, p_2, p_3, p_4 are adjoint variables

Theorem 3.2
Let u_1^*, u_2^* be an optimal control and $M = (M_1, M_2), T = (T_1, T_2)$, be the corresponding state solutions, then there exists $p = (p_1, p_2, p_3, p_4)$ satisfying the adjoint systems

$$
\begin{aligned}
\dot{p}_1 &= -\frac{\partial H}{\partial M_1} = (\alpha_1 T_1 - a) p_1 + a p_2 - \beta_1 T_1 p_3, \\
\dot{p}_2 &= -\frac{\partial H}{\partial M_2} = (\alpha_2 T_2 - b) p_2 + b p_1 - \beta_2 T_2 p_4, \\
\dot{p}_3 &= -\frac{\partial H}{\partial T_1} = (c - \beta_1 M_1) p_3 + \beta_1 p_4 + \alpha_1 M_1 p_1, \\
\dot{p}_4 &= -\frac{\partial H}{\partial T_2} = (d - \beta_2 M_2) p_4 + \beta_2 p_3 + \alpha_2 M_2 p_2.
\end{aligned} \tag{11}
$$

and the transversality condition

$$p_1(T) = p_2(T) = p_3(T) = p_4(T) = 0. \tag{12}$$

While, the optimality control is given by

$$\begin{aligned} u_1^* &= min\left(max(0, \frac{ap_3}{2A}), 1 \right), \\ u_2^* &= min\left(max(0, \frac{bp_4}{2A}), 1 \right). \end{aligned} \tag{13}$$

Proof:

Let $H(t, M, T, u) = \sum_{i=1}^{2}(cM_i + Au_i^2) + p_1\dot{M}_1 + p_2\dot{M}_2 + p_3\dot{T}_1 + p_4\dot{T}_2.$

Where H is the Hamiltonian

By using the Pontryagin maximum principle, there exist adjoint variables $p_i, i = 1,...,4$ satisfying

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial M_1}, \\ \dot{p}_2 &= -\frac{\partial H}{\partial M_2}, \\ \dot{p}_3 &= -\frac{\partial H}{\partial T_1}, \\ \dot{p}_4 &= -\frac{\partial H}{\partial T_2}. \end{aligned} \tag{14}$$

with the transversality conditions $p_1(T) = p_2(T) = p_3(T) = p_4(T) = 0.$ (15)

The optimality conditions is used to derive the optimal control for the species 1 and 2

$$\frac{\partial H}{\partial u_1} = 0, \qquad \frac{\partial H}{\partial u_2} = 0. \tag{16}$$

$$\begin{aligned} 2Au_1 - ap_3 &= 0, \\ 2Au_2 - bp_4 &= 0. \end{aligned} \tag{17}$$

By solving for u_1^* and u_2^* on the interior of the control, then

$$\begin{aligned} u_1^* &= \frac{1}{2A}(ap_3), \\ u_2^* &= \frac{1}{2A}(bp_4). \end{aligned} \tag{18}$$

The adjoint variables P_1, P_2, P_3, P_4 are then obtained as

$$\begin{aligned}
\dot{p}_1 &= (\alpha_1 T_1 - a)p_1 + ap_2 - \beta_1 T_1 p_3 \\
\dot{p}_2 &= (\alpha_2 T_2 - b)p_2 + bp_1 - \beta_2 T_2 p_4 \\
\dot{p}_3 &= (c - \beta_1 M_1)p_3 + \beta_1 p_4 + \alpha_1 M_1 p_1 \quad . \\
\dot{p}_4 &= (d - \beta_2 M_2)p_4 + \beta_2 p_3 + \alpha_2 M_2 p_2
\end{aligned} \tag{19}$$

Using the control bounds to obtain;

$$\begin{aligned}
u_1^* &= min\left(max(0, \frac{ap_3}{2A}), 1 \right) \\
u_2^* &= min\left(max(0, \frac{bp_4}{2A}), 1 \right).
\end{aligned} \tag{20}$$

4. Numerical Implications

The numerical solutions to the system was obtained using the fourth order Runge Kutta method with the use of the MATLAB mathematical package, Lucas(2011). The following parameter values were used, Rafikof et al. (2008) and Aderinto et al. (2013).

a = 0.17, b = 0.116, $\alpha_1 = \alpha_2 = 0.20$, c = d = 0.00017, $\beta_1 = \beta_2 = 0.0085$, $0 \le u_1 \le 1$, $0 \le u_2 \le 1$.

The state equation then becomes

$$\begin{aligned}
\dot{M}_1 &= 0.17M_1 - 0.20M_1T_1 - 0.116M_2 \\
\dot{M}_2 &= 0.116M_2 - 0.20M_2T_2 - 0.17M_1 \\
\dot{T}_1 &= -0.00017T_1 + 0.0085M_1T_1 - 0.0085T_2 - 0.17u_1 \\
\dot{T}_2 &= -0.00017T_2 + 0.0085M_2T_2 - 0.0085T_1 - 0.116u_2.
\end{aligned} \tag{21}$$

u_1, u_2 were taking radomly between 0 and 1 to observe the behavior of the model.

Using different values of $u_1 = 0.60, 0.94, 0.21$ and $u_2 = 0.50, 0.87, 0.19$, the results is presented graphically as in figures 1,2,and 3

Figure 1: Graph of M_1, M_2, T_1, T_2 with u_1 = 0.6, and u_2 = 0.5.

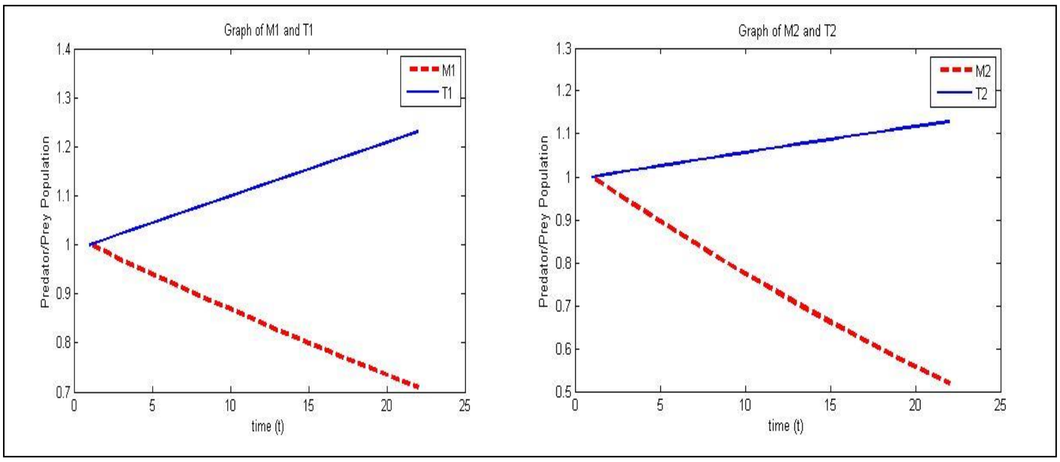


Figure 2: Graph of M_1, M_2, T_1, T_2 with $u_1 = 0.94$, and $u_2 = 0.87$

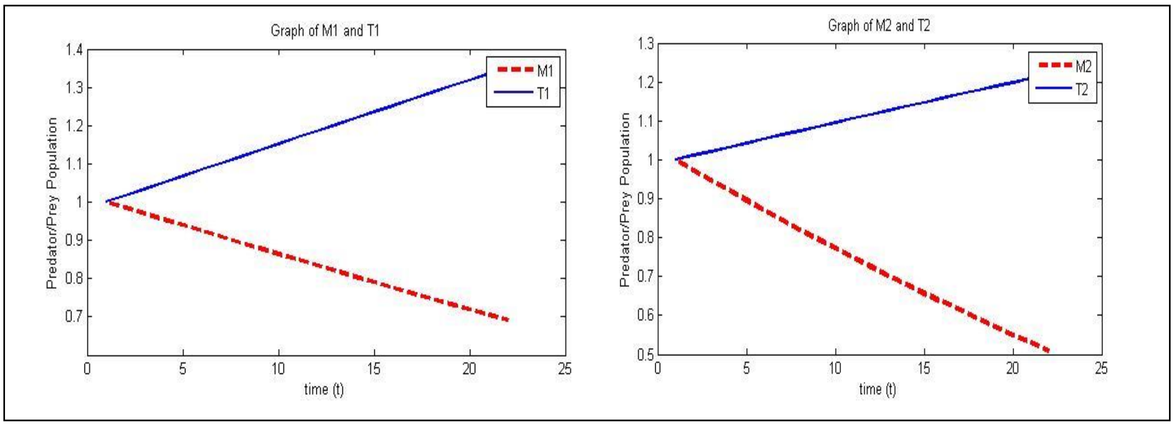
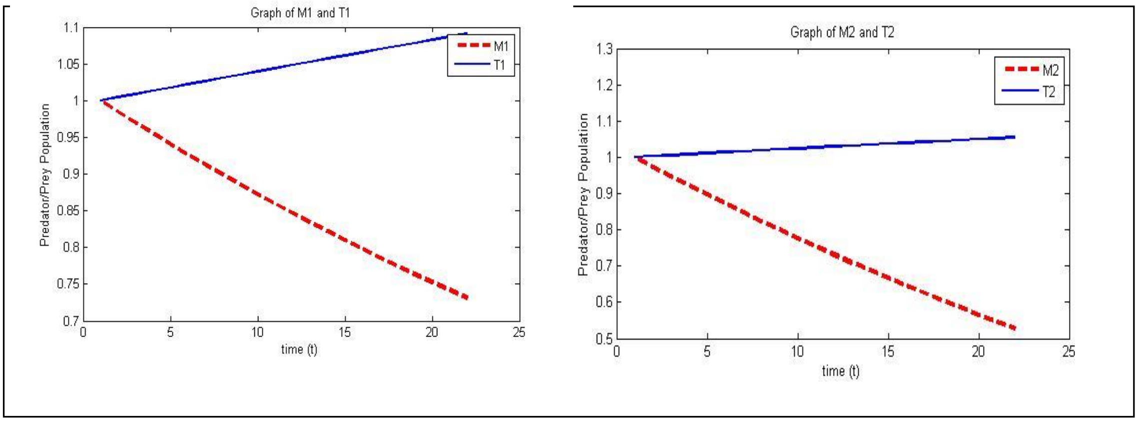


Figure 3: Graph of M_1, M_2, T_1, T_2 with $u_1 = 0.21$, and $u_2 = 0.19$



5. Discussion of the Result

The results obtained from the graphs shows that as the predator population increase, there is decrease in the prey population of the two species. Most especially when u1 =0.6 and u2 =0.5.This shows that the population of the pest can be minimized below injury level using biological control approach.

6. Conclusion

The paper presents the mathematical model of the biological control of two species of cassava pests. The model was analysed using Pontryagin Maximum/Minimum Principle and optimality conditions, the model solution was found to be exist and unique and the optimal effort necessary to reduce the population of the two species was determined. Numerical data were employed to test for the validity of the model using Fourth order Runge Kutta method. And the result obtained shows that pest population of the two species can be minimized to a considerable extent(below injury level) as a result of the introduction of their respective predators.

References

Aderinto, Y. O., Bamigbola, O. M., Jimoh, F. M., Ganiyu, M. A. and Aliu, T. (2013). A Qualitative Study of Biological Pest Control System. *Asian Journal of Mathematics and Statistics*, 43-51.

Andres, L. A., Oatman, E. R., Simpson, R. G., Davis, D. W., Hoyt, S. C., McMurtry, J. A., & Ali Niazee, M. T. (1979). *Re-examination of pest control practices. Biological control and insect pest management.*

Davis, D. W., S.C Hoyt, J.A. McMurtry and M.T AliNiazee (Eds). University of California, Oakland, USA.

Barbu, V. (1994). *Mathematical Methods in Optimization of Differential Systems*. Kluwer Academic Publishers, Netherlands.

Burghes, D. N. and Graham, A. (1987). *An Introduction to Control Theory including Optimal Control*. The Open University, Milton Keynes.

Goh, B. S., Leitman, G. and Vincent, T. L. (1974). The Optimal Control of a Prey - Predator System. *Mathematical Bioscience*, pg 263 - 286.

Herren, H. R. and Neuenschwander, P. (1991). Biological Control of Cassava Pests in Africa. *Annual Review Entomology* 257 - 283.

Lucas, C P. (2011). *Analyzing Predator - Models using Systems of Ordinary Linear Differential Equations*. Southern Illinois University, Carbondale.

Michael, M. O. (2008). *Essential Biology*. Tonad Publishers.

Mutisya, D. L., El-Banhawy, E. M., Kariuki, C. W., & Khamala, C. P. M. (2014). Typhlodromalus aripo De Leon (Acari: Phytoseiidae) development and reproduction on major cassava pests at different temperatures and humidities: an indication of enhanced mite resilience. *Acarologia*, 54(4), 395-407.

Picart, D., Ainseba, B., Fabio, M. (2011). Optimal Control Problem on Insect Pest Populations. *Applied Mathematics Letters* 24, pp. 1160 - 1164.

Rafikov, M. and Balthazzar, J.M. (2005). Optimal pest control problem in population dynamics. *Applied Math., Comput.*, 24: 65 - 81.

Rafikov, M., Balthazzar, J. M. and Bremen, (2008). Mathematical Modelling and Control of Population Systems: Applications in biological pest control. *Applied Math. Compu.*, 200: 557 - 573.